International Journal of Theoretical Physics, Vol. 30, No. 11, 1991

Lipschitz Quasistability of Impulsive Systems of Differential Equations

G. K. Kulev[1] and D. D. Bainov[1]

Received August 12, 1990

A suitable comparison lemma is used to obtain sufficient conditions for uniform Lipschitz quasistability of an arbitrary solution of an impulsive system of differential equations with unfixed moments of impulse effect. The results are applied to finding conditions for uniform Lipschitz quasistability for linear impulsive systems with unfixed moments of impulse effect.

1. INTRODUCTION

Impulsive systems of differential equations are useful for the mathematical simulation of numerous real processes and phenomena studied in biology, physics, technology, etc. Such processes and phenomena are characterized by the fact that at certain moments of their evolution they undergo rapid changes. That is why in their mathematical simulation it is convenient to neglect the duration of these changes and assume that such processes and phenomena change their state momentarily, by jumps.

Moreover, the mathematical theory of impulsive systems of differential equations is much richer in problems in comparison with the corresponding theory of ordinary differential equations without impulses. That is why in recent years the study of such systems has been very intensive [see Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), and Samoilenko and Perestyuk (1987), and references therein].

In the present paper the notion of *uniform Lipschitz stability* of an arbitrary solution of an impulsive system of differential equations with unfixed moments of impulse effect is introduced. By means of a suitable comparison method, sufficient conditions for uniform Lipschitz stability of a given solution of such a system are found. Since in this case the moments

[1]Plovdiv University "Paissii Hilendarski," Plovdiv, Bulgaria.

0020-7748/91/0800-1151$06.50/0

of impulse effect are different for the different solutions, for impulsive systems of this type there is no continuous dependence, uniform on a finite interval, of their solutions on the initial conditions (Samoilenko and Perestyuk, 1987, §3; Lakshmikantham *et al.*, 1989, §2.3). That is why for these systems one cannot speak of Lipschitz stability of an arbitrary solution in the usual sense (Dannan and Elaydi, 1986). In relation to this, in this paper the sense in which the notion of uniform Lipschitz stability of a given solution of an impulsive system of differential equations with unfixed moments of impulse effect should be understood is made precise by introducing the notion of *uniform Lipschitz quasistability*.

2. PRELIMINARY NOTES AND DEFINITIONS

Consider the impulsive system of differential equations

$$\dot{x}=f(t, x), \quad t\neq \tau_k(x); \qquad \Delta x|_{t=\tau_k(x)}=I_k(x) \tag{1}$$

where $x\in\mathbb{R}^n, f\colon \mathbb{R}_+\times\Omega\to\mathbb{R}^n,\ \tau_k\colon\Omega\to\mathbb{R}_+,\ I_k\colon\Omega\to\mathbb{R}^n$,

$$\Delta x|_{t=\tau_k(x)}=x(t+0)-x(t-0)$$

$\mathbb{R}_+=[0, \infty)$, and Ω is an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with an arbitrary norm $|\cdot|$.

A detailed description of impulsive systems of the form (1) can be found in Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), and Samoilenko and Perestyuk (1987).

Let $x_0(t)=x_0(t;\, t_0, y_0)$ be a solution of system (1) satisfying the initial condition $x_0(t_0+0)=y_0$ and which is defined on the interval (t_0, ∞). Let $t=t_k$, $k=1, 2, \ldots$, be the moments at which the integral curve of this solution meets the hypersurfaces

$$\sigma_k=\{(t, x)\in\mathbb{R}_+\times\Omega\colon t=\tau_k(x)\}$$

i.e., $t_k=\tau_k(x_0(t_k))$, $k=1, 2, \ldots$.

We shall say that conditions (A) are met if the following conditions hold:

A1. $f\in C[\mathbb{R}_+\times\Omega, \mathbb{R}^n]$ and $|f(t, x)|\leq L$ for $(t, x)\in\mathbb{R}_+\times\Omega$.
A2. $I_k\in C[\Omega, \mathbb{R}^n]$, $k=1, 2, \ldots$.
A3. $\tau_k\in C^1[\Omega, \mathbb{R}_+]$, $k=1, 2, \ldots$.
A4. $0<\tau_1(x)<\tau_2(x)<\cdots<\tau_k(x)<\cdots$ and $\lim_{k\to\infty}\tau_k(x)=\infty$ for $x\in\Omega$.

We shall say that condition (B) is met if the following condition holds:

B. The integral curve of each solution of system (1) meets each of the hypersurfaces σ_k at most once.

When condition (B) is met, we say that for system (1) the phenomenon of "beating" is absent. Sufficient conditions for absence of the phenomenon of "beating" are given in Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), and Samoilenko and Perestyuk (1987).

Definition 1. The solution $x_0(t)$ of system (1) is said to be *uniformly Lipschitz quasistable* if

$$
\begin{gathered}
(\exists M>0)(\forall \eta>0)(\exists \delta=\delta(\eta)>0)(\forall x_0\in\Omega,\ |x_0-y_0|<\delta)\\
(\forall t_0\in\mathbb{R}_+)(\forall t>t_0,\ |t-t_k|>\eta,\ k=1,2,\ldots):\\
|x(t;t_0,x_0)-x_0(t)|\le M|x_0-y_0|
\end{gathered}
$$

3. COMPARISON LEMMA

Since the moments of impulse effect for the different solutions of system (1) are different, there are difficulties in the estimation of the difference of two different solutions of this system. In order to overcome these difficulties, we shall use a suitable comparison lemma.

Consider a scalar impulsive differential equation of the form

$$
\begin{cases}
\dot u=g(t,u), & t\bar{\in}(\underline{t}_k,\bar t_k],\quad k=1,2,\ldots\\
u(\bar t_k+0)=\psi_k(u(\underline{t}_k)), & k=1,2,\ldots\\
u(t_0+0)=u_0
\end{cases}
\tag{2}
$$

where

$$
g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R},\qquad \psi_k:\mathbb{R}\to\mathbb{R},\qquad k=1,2,\ldots
$$

$$
0\le t_0<\underline{t}_1\le\bar t_1<\underline{t}_2\le\bar t_2<\cdots<\underline{t}_k\le\bar t_k<\cdots\quad\text{and}\quad \lim_{x\to\infty}\underline{t}_k=\infty \tag{3}
$$

The solution $u(t;t_0,u_0)$ of equation (2) is determined in the following way:

$$
u(t;t_0,u_0)=\begin{cases}
u_0(t;t_0,u_0), & t_0<t\le\underline{t}_1\\
u_1(t;\bar t_1,u_1^+), & \bar t_1<t<\underline{t}_2\\
\cdots\\
u_k(t;\bar t_k,u_k^+), & \bar t_k<t<\underline{t}_{k+1}\\
\cdots
\end{cases}
$$

where $u_k(t; \bar{t}_k, u_k^+)$, $k=1, 2, \ldots$, is a solution of the equation $\dot{u}=g(t, u)$, for which $u_k(\bar{t}_k; \bar{t}_k, u_k^+)=u_k^+$ and $u_k^+ = \psi_k(u_{k-1}(\underline{t}_k; \bar{t}_{k-1}, u_{k-1}^+))$, $k=2, 3, \ldots$, and $u_0(t; t_0, u_0)$ is a solution of $\dot{u}=g(t, u)$, for which $u_0(t_0; t_0, u_0)=u_0$ and $u_1^+ = \psi_1(u_0(\underline{t}_1; t_0, u_0))$.

The maximal solution $r(t; t_0, u_0)$ of equation (1) is defined in an analogous way.

Lemma 1. (Lakshmikantham *et al.*, 1989, Theorem 1.6.1). Let the following conditions hold:

1. The function $m: \mathbb{R}_+ \to \mathbb{R}$ is piecewise continuous with points of discontinuity of the first kind $t=\underline{t}_k$ and $t=\bar{t}_k$ at which it is continuous from the left and for which relations (3) are valid.

2. For $k=1, 2, \ldots$ the inequalities

$$\begin{aligned} &D^+m(t) \le g(t, m(t)), \qquad t\bar{\in}(\underline{t}_k, \bar{t}_k] \\ &m(\bar{t}_k+0) \le \psi_k(m(\underline{t}_k)) \\ &m(t_0+0) \le u_0 \end{aligned}$$

hold, where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $\psi_k \in C[\mathbb{R}, \mathbb{R}]$, $\psi_k(u)$ is nondecreasing in u, and

$$D^+m(t) = \limsup_{h\to 0^+} (1/h)[m(t+h)-m(t)]$$

3. The maximal solution $r(t; t_0, u_0)$ of equation (1) is defined in

$$\mathscr{I} = (t_0, \infty) \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k].$$

Then $m(t) \le r(t; t_0, u_0)$ for $t \in \mathscr{I}$.

4. MAIN RESULTS

Theorem 1. Let the following conditions be fulfilled:

1. Conditions (A) and (B) hold.

2. For $(t, x) \in \mathbb{R}_+ \times \Omega$, $t \ne t_k$, $t \ne \tau_k(x)$, $k=1, 2, \ldots$, the following inequality is valid:

$$[x-x_0(t), f(t, x)-f(t, x_0(t))]_+ \le g(t, |x-x_0(t)|)$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ and

$$[x, y]_+ = \limsup_{h\to 0^+} (1/h)(|x+hy|-|x|)$$

3. For $x \in \Omega$ and $k=1, 2, \ldots$ the following inequalities are valid:

$$|x-x_0(t_k)+I_k(x)-I_k(x_0(t_k))| \le \gamma_k(|x-x_0(t)|)$$

where $\gamma_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ and $\gamma_k(u)$ is nondecreasing in u.

4. For $(t, x)\in\mathbb{R}_+\times\Omega$, $k=1, 2, \ldots$, the following inequalities hold:

$$\frac{\partial\tau_k(x)}{\partial x}f(t, x)\leq 0$$

5. For $x, y\in\Omega$ and $k=1, 2, \ldots$ the inequalities

$$|\tau_k(x)-\tau_k(y)|\leq\beta|x-y|$$

hold, where $0<\beta=\text{const}$.

6. There exist constants $M>0$ and $\delta_1>0$ such that for any $x\in\Omega$ and for any solution $u(t; t_0, u_0)$ of the scalar impulsive differential equation

$$\begin{cases}\dot{u}=g(t, u), & t\bar{\in}(\underline{\xi}_k, \bar{\xi}_k], \quad k=1, 2, \ldots\\ u(\bar{\xi}_k+0)=\psi_k(u(\underline{\xi}_k)), & k=1, 2, \ldots\\ u(t_0+0)=u_0\end{cases}\tag{4}$$

where

$$\psi_k(u)=\gamma_k((1+L\beta)u)+L\beta u$$

$$\underline{\xi}_k=\min(t_k, \tau_k(x)), \qquad \bar{\xi}_k=\max(t_k, \tau_k(x))$$

the following inequality holds:

$$u(t; t_0, u_0)\leq Mu_0 \qquad \text{for} \quad 0\leq u_0<\delta_1, \quad t\in(t_0, \infty)\Big\backslash\bigcup_{k=1}^{\infty}(\underline{\xi}_k, \bar{\xi}_k]\tag{5}$$

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

Proof. Let $\eta>0$ be given. Choose $\delta=\delta(\eta)=\min(\delta_1, \eta/(2M\beta+1))$.

Let $x(t)=x(t; t_0, x_0)$ be any solution of (1) for which $|x_0-y_0|<\delta$ and let $t_k^*=\tau_k(x(t_k^*))$, $k=1, 2, \ldots$, be the moments of impulse effect for this solution.

Set $m(t)=|x(t)-x(t_0)|$, $u_0=|x_0-y_0|$, and $\underline{t}_k=\min(t_k, t_k^*)$,

$$\bar{t}_k=\max(t_k, t_k^*)$$

Then $m(t_0+0)=u_0$. From condition 2 it follows that

$$D^+m(t)\leq g(t, m(t)), \qquad t\bar{\in}(\underline{t}_k, \bar{t}_k], \qquad k=1, 2, \ldots\tag{6}$$

We shall estimate $m(\bar{t}_k+0)=|x(\bar{t}_k+0)-x_0(\bar{t}_k+0)|$ for an arbitrary positive integer k.

In the case when $\bar{t}_k = t_k^*$ and $\underline{t}_k = t_k$, using conditions 3 and A1, we obtain

$$\begin{aligned} m(\bar{t}_k+0) &= |x(\bar{t}_k) + I_k(x(\bar{t}_k)) - x_0(\bar{t}_k)| \\ &\leq |x(\bar{t}_k) - x_0(\underline{t}_k) + I_k(x(\bar{t}_k)) - I_k(x_0(\underline{t}_k))| \\ &\quad + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x_0(s))|\, ds \\ &\leq \gamma_k(|x(\bar{t}_k) - x_0(\underline{t}_k)|) + L(\bar{t}_k - \underline{t}_k) \end{aligned}$$

On the other hand,

$$\begin{aligned} |x(\bar{t}_k) - x_0(\underline{t}_k)| &\leq |x(\underline{t}_k) - x_0(\underline{t}_k)| + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x(s))|\, ds \\ &\leq m(\underline{t}_k) + L(\bar{t}_k - \underline{t}_k) \end{aligned}$$

From condition 4 it follows that $\tau_k(x(\bar{t}_k)) \leq \tau_k(x(\underline{t}_k))$. Then from condition 5 we obtain

$$\begin{aligned} 0 \leq \bar{t}_k - \underline{t}_k &= \tau_k(x(\bar{t}_k)) - \tau_k(x_0(\underline{t}_k)) \leq \tau_k(x(\underline{t}_k)) - \tau_k(x_0(\underline{t}_k)) \\ &\leq \beta |x(\underline{t}_k) - x_0(\underline{t}_k)| = \beta m(\underline{t}_k) \end{aligned} \tag{7}$$

Hence

$$m(\bar{t}_k+0) \leq \gamma_k((1+L\beta)u(\underline{t}_k)) + L\beta u(\underline{t}_k) = \psi_k(u(\underline{t}_k)) \tag{8}$$

In the case when $\bar{t}_k = t_k$ and $\underline{t}_k = t_k^*$, we again use conditions 3 and A1 and obtain

$$\begin{aligned} m(\bar{t}_k+0) &\leq |x(\underline{t}_k) - x_0(\bar{t}_k) + I_k(x(\underline{t}_k)) - I_k(x_0(\bar{t}_k))| \\ &\quad + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x(s))|\, ds \\ &\leq \gamma_k(|x(\underline{t}_k) - x_0(\bar{t}_k)|) + L(\bar{t}_k - \underline{t}_k) \end{aligned}$$

Moreover,

$$|x(\underline{t}_k) - x_0(\bar{t}_k)| \leq m(\underline{t}_k) + L(\bar{t}_k - \underline{t}_k)$$

and from conditions A4 and A5 we obtain

$$\begin{aligned} 0 \leq \bar{t}_k - \underline{t}_k &= \tau_k(x_0(\bar{t}_k)) - \tau_k(x(\underline{t}_k)) \leq \tau_k(x_0(\underline{t}_k)) - \tau_k(x(\underline{t}_k)) \\ &\leq \beta |x_0(\underline{t}_k) - x(\underline{t}_k)| = \beta m(\underline{t}_k) \end{aligned} \tag{9}$$

Hence

$$m(\bar{t}_k+0)\leq \gamma_k((1+L\beta)m(\underline{t}_k))+L\beta m(\underline{t}_k)=\psi_k(m(\underline{t}_k)) \tag{10}$$

Inequalities (6), (8), and (10) show that the conditions of Lemma 1 are fulfilled. Then

$$|x(t)-x_0(t)|=m(t)\leq r(t;\, t_0,\, |x_0-y_0|) \tag{11}$$

for $t\in(t_0,\,\infty)\backslash\bigcup_{k=1}^{\infty}(\underline{t}_k,\,\bar{t}_k]$, where $r(t;\,t_0,\,|x_0-y_0|)$ is the maximal solution of (4) for $\xi_k=\underline{t}_k$ and $\bar{\xi}_k=\bar{t}_k$, $k=1, 2, \ldots$.

From (5) and (11) it follows that

$$|x(t)-x_0(t)|\leq M|x_0-y_0| \qquad \text{for} \quad t\in(t_0,\,\infty)\Big\backslash\bigcup_{k=1}^{\infty}(\underline{t}_k,\,\bar{t}_k] \tag{12}$$

Moreover, from (7) and (9) and the choice of δ we obtain

$$0\leq\bar{t}_k-\underline{t}_k\leq\beta|x(\underline{t}_k)-x_0(\underline{t}_k)|\leq\beta M|x_0-y_0|\leq M\beta\delta<\eta/2$$

Hence

$$|x(t)-x_0(t)|\leq M|x_0-y_0|$$

for $|x_0-y_0|<\delta$, $t>t_0\geq 0$, $|t-t_k|>\eta$, $k=1, 2, \ldots$.

Theorem 1 is proved. ■

Corollary 1. Let the following conditions be fulfilled:

1. Conditions (A) and (B) hold.
2. For $(t, x)\in S(x_0, \rho)=\{(t, x)\in\mathbb{R}_+\times\mathbb{R}^n;\ |x-x_0(t)|<\rho\}$ $(\rho>0)$, $t\neq t_k$, $t\neq\tau_k(x)$, $k=1, 2, \ldots$, the following inequality is valid:

$$[x-x_0(t), f(t, x)-f(t, x_0(t))]_+\leq 0$$

3. For $x\in S(\rho)=\bigcup_{t\in\mathbb{R}_+}\{x\in\mathbb{R}^n\colon\ |x-x_0(t)<\rho\}$ and $k=1, 2, \ldots$, the following inequalities are valid:

$$|x-x_0(t_k)+I_k(x)-I_k(x_0(t_k))|\leq\gamma_k|x-x_0(t_k)|; \qquad |I_k(x)|\leq\rho/3$$

where $\gamma_k\geq 0$ are constants.

4. For $(t, x)\in S(x_0, \rho)$ and $k=1, 2, \ldots$, the following inequalities are valid:

$$\frac{\partial\tau_k(x)}{\partial x}f(t, x)\leq 0$$

5. For $x, y\in S(\rho)$ and $k=1, 2, \ldots$, the following inequalities are valid:

$$|\tau_k(x)-\tau_k(y)|\leq\beta|x-y|$$

where $0<\beta=\text{const}$.

6. The infinite product $\prod_{k=1}^{\infty} [\gamma_k + (1+\gamma_k)L\beta]$ is convergent.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

Theorem 2. Let the following conditions hold:

1. Conditions 1–3 of Theorem 1 are fulfilled.
2. For $x, y \in \Omega$ and $k = 1, 2, \ldots$, the inequalities

$$|\tau_k(x) - \tau_k(y)| \leq \beta_k |x - y|$$

hold, where $\beta_k \geq 0$ are constants.

3. For $k = 1, 2, \ldots$, the following inequalities are valid:

$$L\beta_k < 1, \qquad \beta_k(1 - L\beta_k)^{-1} \leq \beta$$

where $0 < \beta = \text{const}$.

4. There exist constants $M > 0$ and $\delta_1 > 0$ such that for any $x \in \Omega$ and for any solution $u(t; t_0, u_0)$ of equation (4) for which

$$\psi_k(u) = \gamma_k((1 - L\beta_k)^{-1}u) + L\beta_k(1 - L\beta_k)^{-1}u$$

inequality (5) is valid.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

The proof of Theorem 2 is analogous to the proof of Theorem 1.

Corollary 2. Let the following conditions be satisfied:

1. Conditions 1–3 of Corollary 1 hold.
2. Conditions 2 and 3 of Theorem 2 hold.
3. The infinite product $\prod_{k=1}^{\infty} (\gamma_k + L\beta_k)(1 - L\beta_k)^{-1}$ is convergent.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

Theorem 3. Let the conditions of Theorem 1 hold, condition 2 being replaced by the following condition:

2a. For $(t, x) \in \mathbb{R}_+ \times \Omega$, $t \neq t_k$, $t \neq \tau_k(x)$, $k = 1, 2, \ldots$, the following inequality is valid:

$$|x - x_0(t) + h(f(t, x) - f(t, x_0(t)))| \leq |x - x_0(t)| + hg(t, |x - x_0(t)|) + \varepsilon(h)$$

where $h > 0$ is small enough and $\varepsilon(h)/h \to 0$ as $h \to 0$.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz stable.

The proof of Theorem 3 is analogous to the proof of Theorem 1. We use that from condition 2a there follow the inequalities

$$
\begin{aligned}
D^+m(t) &= \limsup_{h\to 0^+}(1/h)[|x(t+h)-x_0(t+h)|-|x(t)-x_0(t)|] \\
&\leq \limsup_{h\to 0^+}(1/h)[|x(t+h)-x_0(t+h)|+\varepsilon(h) \\
&\quad -|x(t)-x_0(t)-h(f(t,x(t))-f(t,x_0(t)))|] \\
&\leq \limsup_{h\to 0^+}\varepsilon(h)/h+\limsup_{h\to 0^+}(1/h)|x(t+h)-x(t) \\
&\quad -x_0(t+h)+x_0(t)-f(t,x(t))+f(t,x_0(t))|=0
\end{aligned}
\tag{13}
$$

Corollary 3. Let the conditions of Corollary 1 hold, condition 2 being replaced by the following condition:

2b. For $(t, x)\in S(x_0, \rho)$, $t\neq t_k$, $t\neq \tau_k(x)$, and $k=1, 2, \ldots$, the following inequality is valid:

$$|x-x_0(t)+h(f(t,x)-f(t,x_0(t)))|\leq|x-x_0(t)|+\varepsilon(h)$$

where $h>0$ is small enough and $\varepsilon(h)/h\to 0$ as $h\to 0$.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

Theorem 4. Let the conditions of Theorem 2 hold, condition 2 of Theorem 1 being replaced by condition 2a.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

The proof of Theorem 4 is analogous to the proof of Theorem 1. Inequalities (13) are used.

Corollary 4. Let the conditions of Corollary 2 be fulfilled, condition 2 of Corollary 1 being replaced by condition 2b.

Then the solution $x_0(t)$ of system (1) is uniformly Lipschitz quasistable.

5. APPLICATIONS

Application 1. Consider the linear impulsive system

$$\dot{x}=Ax, \qquad t\neq\tau_k(x); \qquad \Delta x|_{t=\tau_k(x)}=B_kx \tag{14}$$

where A and B_k, $k=1,2,\ldots$, are constant $n\times n$ matrices and $\tau_k\colon \mathbb{R}^n\to\mathbb{R}_+$ satisfy conditions A3, A4, and (B).

Consider system (14) in the domain $t\geq 0$, $|x|\leq L/\|A\|$, where $\|A\|=\sup\{|Ax|: |x|\leq 1\}$, $L>0$.

Let $x(t)=x(t; t_0, x_0)$ be an arbitrary solution of (14) defined in the interval (t_0, ∞) and let $t=t_k$, $k=1,2,\ldots$, be its moments of impulse effect.

It is immediately verified that

$$[x-x_0(t), A(x-x_0(t))]_+ \leq \mu(A)|x-x_0(t)|, \qquad t>t_0$$

where Lozinskii's "logarithmic norm" $\mu(A)$ of the matrix A is defined by the equality

$$\mu(A)=\lim_{h\to 0^+}\sup(1/h)(\|E+hA\|-1)$$

(E is the unit $n\times n$ matrix).

Consider the following conditions:

(i) $\mu(A)\leq 0$.

(ii) $\|E+B_k\|\leq\gamma_k$, $k=1, 2, \ldots$, $\beta_k\geq 0$.

(iii) $[\partial\tau_k(x)/\partial x]Ax\leq 0$, $k=1, 2, \ldots$, $|x|\leq L/\|A\|$.

(iv) $|\tau_k(x)-\tau_k(y)|\leq\beta_k|x-y|$, $|x|\leq L/\|A\|$, $|y|\leq L/\|A\|$, $\beta_k\geq 0$, $k=1, 2, \ldots$.

(v) $\beta_k\leq\beta$ ($\beta>0$), $k=1, 2, \ldots$.

(vi) The product $\prod_{k=1}^{\infty}(\gamma_k+(1+\gamma_k)L\beta)$ is convergent.

(vii) $L\beta_k<1$, $\beta_k(1-L\beta_k)^{-1}\leq\beta$ ($\beta>0$), $k=1, 2, \ldots$.

(viii) $\prod_{k=1}^{\infty}(\gamma_k+L\beta_k)(1-L\beta_k)^{-1}$ is convergent.

For an arbitrary $x\in\mathbb{R}^n$, $|x|\leq L/\|A\|$, consider the impulsive differential equation

$$\begin{cases}\dot{u}=\mu(A)u, & t\bar{\in}(\underline{t}_k, \bar{t}_k], \quad k=1, 2, \ldots\\ u(\bar{t}_k+0)=(\gamma_k+(1+\gamma_k)L\beta)u(\underline{t}_k), & k=1, 2, \ldots\end{cases}$$

where $\underline{t}_k=\min(t_k, \tau_k(x))$, $\bar{t}_k=\max(t_k, \tau_k(x))$, $k=1, 2, \ldots$, whose solution is determined by means of the equality

$$u(t; t_0, u_0)=u_0\left[\prod_{j=1}^{k}(\gamma_j+(1+\gamma_j)L\beta)\right]\exp\left[\mu(A)(t-t_0-\sum_{j=1}^{k}(\bar{t}_j-\underline{t}_j)\right]$$

for $\bar{t}_k<t\leq\underline{t}_{k+1}$, $k=1, 2, \ldots$.

Let conditions (i)–(vi) hold. Then, applying Theorem 1 (or Corollary 1), we obtain that the solution $x_0(t)$ of (14) is uniformly Lipschitz quasistable.

In an analogous way it is proved that if conditions (i), (ii), (iv), (vii), and (viii) hold, then the conditions of Theorem 2 (or of Corollary 2) are satisfied. Hence the solution $x_0(t)$ of (14) is uniformly Lipschitz quasistable.

Application 2. Consider the linear impulsive system

$$\dot{x}=A(t)x, \quad t\neq\tau_k(x); \qquad \Delta x|_{t=\tau_k(x)}=B_kx \tag{15}$$

where $A(t)$ is continuous in $\mathbb{R}_+$ $n\times n$ matrix for which $\|A(t)\|\leq M$ $(M>0)$, for $t\in\mathbb{R}_+$, B_k, $k=1,2,\ldots$, are constant $n\times n$ matrices, $|x|\leq L/M$ $(L>0)$, and τ_k: $\mathbb{R}^n\to\mathbb{R}_+$ satisfy conditions A3, A4, and (B).

Let $x_0(t)=x_0(t;t_0,x_0)$, $t_0\in\mathbb{R}_+$, $|x_0|<L/M$, be an arbitrary solution of (15) defined in (t_0,∞) and let $t=t_k$, $k=1,2,\ldots$, be its moments of impulse effect.

Consider the condition

$$\text{(ia)}\quad \limsup_{t\to\infty}\int_{t_0}^{t}\mu(A(s))\,ds<\infty$$

For an arbitrary $x\in\mathbb{R}^n$, $|x|\leq L/M$, consider the impulsive differential equation

$$\begin{cases}\dot{u}=\mu(A(t))u, & t\bar{\in}(\underline{t}_k,\bar{t}_k], \quad k=1,2,\ldots\\ u(\bar{t}_k+0)=(\gamma_k+(1+\gamma_k)L\beta)u(\underline{t}_k), & k=1,2,\ldots\end{cases}$$

where $\underline{t}_k=\min(t_k,\tau_k(x))$, $\bar{t}_k=\max(t_k,\tau_k(x))$, $k=1,2,\ldots$, whose solution is determined by means of the equality

$$u(t;t_0,u_0)$$
$$=u_0\left[\prod_{j=1}^{k}(\gamma_k+(1+\gamma_k)L\beta)\right]\exp\left[\int_{t_0}^{t}\mu(A(s))\,ds-\sum_{j=1}^{k}\int_{\underline{t}_j}^{\bar{t}_j}\mu(A(s))\,ds\right]$$

for $\bar{t}_k<t\leq\underline{t}_{k+1}$, $k=1,2,\ldots$.

If conditions (ia) and (ii)–(vi) hold, then the conditions of Theorem 1 (or of Corollary 1) are satisfied. Hence the solution $x_0(t)$ of (15) is uniformly Lipschitz quasistable.

Application 3. For system (1) let the following conditions hold:

(a) Conditions 1 and 3–5 of Theorem 1 are satisfied.

(b) For $(t,x)\in\mathbb{R}_+\times\Omega$, $t\neq t_k$, $t\neq\tau_k(x)$, $k=1,2,\ldots$, the following inequality is valid:

$$[x-x_0(t),f(t,x)-f(t,x_0(t))]_+\leq\rho(t)\Phi(|x-x_0(t)|)$$

where ρ, $\Phi\in C[\mathbb{R}_+,\mathbb{R}_+]$, $\Phi(u)$ is strictly increasing in u, and $\Phi(0)=0$.

(c) For any $x\in\Omega$ and any $\sigma>0$ the following inequality is valid:

$$\int_{\tau_k(x)}^{\tau_{k+1}(x)}\rho(s)\,ds+\int_{\sigma}^{\psi_k(\sigma)}\frac{ds}{\Phi(s)}\leq 0,\qquad k=1,2,\ldots$$

where $\psi_k(u)=\gamma_k((1+L\beta)u)+L\beta u$.

For any $x \in \Omega$ consider the impulsive differential equation

$$\begin{cases} \dot{u} = \rho(t)\Phi(u), & t \bar{\in} (\underline{t}_k, \bar{t}_k], \quad k = 1, 2, \ldots \\ u(\bar{t}_k + 0) = \psi_k(u(\underline{t}_k)), & k = 1, 2, \ldots \end{cases}$$

where $\underline{t}_k = \min(t_k, \tau_k(x))$, $\bar{t}_k = \max(t_k, \tau_k(x))$, $k = 1, 2, \ldots$

From condition (c) it follows that condition 6 of Theorem 1 holds. Hence the solution $x_0(t)$ of (1) is uniformly Lipschitz quasistable.

In the same way, using Theorems 2–4, one can obtain conditions analogous to conditions (a)–(c), under which the solution $x_0(t)$ of (1) is uniformly Lipschitz quasistable.

ACKNOWLEDGMENT

The present investigation is supported by the Bulgarian Ministry of Culture, Science and Education of the People's Republic of Bulgaria under Grant 61.

REFERENCES

Bainov, D. D., and Simeonov, P. S. (1989). *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood Limited.

Dannan, F. M., and Elaydi, S. (1986). *Journal of Mathematical Analysis and Applications*, **113**, 562–577.

Lakshmikantham, V., Bainov, D. D., and Simeonov, P. S. (1989). *Theory of Impulsive Differential Equations*, World Scientific, Singapore.

Samoilenko, A. M., and Perestyuk, N. A. (1987). *Differential Equations with Impulse Effect*, Višča Škola, Kiev (in Russian).